Bounds of discriminants of number fields

Pei-Chu Hu *

1 Basic notations

1.1 Number fields

A complex number α will be called *algebraic* if it is algebraic over the field $\mathbb{Q}$ of rational numbers, that is, it satisfies a non-zero polynomial equation with coefficients in $\mathbb{Q}$. We let $\bar{\mathbb{Q}}$ denote the set of algebraic numbers, which, in fact, is a field.

An element α of the field $\mathbb{C}$ of complex numbers is said to be *integral* over the ring $\mathbb{Z}$ of integral numbers if it satisfies a monic equation over $\mathbb{Z}$:

$$\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0, \tag{1}$$

where $a_i \in \mathbb{Z}$ for $i = 1, ..., n$. The integral elements $\bar{\mathbb{Z}} \subset \mathbb{C}$ over $\mathbb{Z}$ are called *algebraic integers*. Then we have

$$\mathbb{Z} = \bar{\mathbb{Z}} \cap \mathbb{Q}.$$

Remark. The whole field $\bar{\mathbb{Q}}$ is not as interesting, for us, as certain of its subfields. The trouble with $\bar{\mathbb{Q}}$ is that the dimension $[\bar{\mathbb{Q}} : \mathbb{Q}]$ of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$ is not finite.

We define a *number field* to be a subfield κ of $\mathbb{C}$ such that the dimension $[\kappa : \mathbb{Q}]$ of κ over $\mathbb{Q}$ is finite. This implies that every element of κ is algebraic, and hence $\kappa \subseteq \bar{\mathbb{Q}}$. The number $[\kappa : \mathbb{Q}]$ is called the degree of κ.

1.2 Places

Let κ be a number field of degree n. An equivalence class of non-trivial absolute values on κ is called a *place* of κ. A place of κ is called *non-Archimedean* or *finite* (resp., *Archimedean* or *infinite*) if its absolute value is non-Archimedean (resp., Archimedean). Usually, let M_κ^0 (resp. M_κ^∞) be all finite (resp. infinite) places of κ, and set

$$M_\kappa = M_\kappa^0 \cup M_\kappa^\infty. \tag{2}$$

To ease notation, we frequently write the absolute values corresponding to a place v of κ as $|\cdot|_v$.

For each $v \in M_\kappa$, there exists a positive real number n_v such that M_κ satisfies the *product formula with multiplicities* n_v

$$\prod_{v \in M_\kappa} |x|_v^{n_v} = 1 \tag{3}$$

for all $x \in \kappa_* = \kappa - \{0\}$. When we deal with a fixed set of multiplicities n_v, then we write for convenience

$$\|x\|_v = |x|_v^{n_v} \tag{4}$$

which is called normalization of $|\cdot|_v$, so that the product formula reads

$$\prod_{v \in M_\kappa} \|x\|_v = 1. \tag{5}$$

Let r_1 be the number of embeddings of κ into $\mathbb{R}$, and let $2r_2$ be the number of embeddings of κ into $\mathbb{C}$, whose image is not contained in $\mathbb{R}$. Then

$$r_1 + 2r_2 = n = [\kappa : \mathbb{Q}], \tag{6}$$

and κ has $r_1 + r_2$ pairwise inequivalent Archimedean absolute values, that is,

$$\# M_\kappa^\infty = r_1 + r_2. \tag{7}$$

In other words, r_1, r_2 are the number of real, complex places respectively.

1.3 Discriminants of number fields

Let κ be a *number field* of degree $n = [\kappa : \mathbb{Q}]$. Take $\alpha \in \kappa$. Then α induces a $\mathbb{Q}$-linear mapping

$$\mathbf{A}_\alpha : \kappa \longrightarrow \kappa \tag{8}$$

defined by $\mathbf{A}_\alpha(x) = \alpha x$. Let $\{w_1, ..., w_n\}$ be a base of κ over $\mathbb{Q}$. Then we can write

$$\mathbf{A}_\alpha(w_i) = \alpha w_i = \sum_{j=1}^{n} a_{ij} w_j \tag{9}$$

for some $a_{ij} \in \mathbb{Q}$. The characteristic polynomial

$$\chi_\alpha(x) = \det(xI - A_\alpha) \tag{10}$$

of the matrix form $A_\alpha = (a_{ij})$ of $\mathbf{A}_\alpha$ is called the *field polynomial* of α. The field polynomial χ_α is independent of the base $\{w_1, ..., w_n\}$ selected for κ over $\mathbb{Q}$. Obviously, α is a root of its field polynomial.

By using field polynomial χ_α of α, we can define respectively the *norm* $\mathbf{N}_{\kappa/\mathbb{Q}}(\alpha)$ and the *trace* $\mathbf{Tr}_{\kappa/\mathbb{Q}}(\alpha)$ of α over $\mathbb{Q}$ by

$$\chi_\alpha(x) = x^n - \mathbf{Tr}_{\kappa/\mathbb{Q}}(\alpha)x^{n-1} + \cdots + (-1)^n\mathbf{N}_{\kappa/\mathbb{Q}}(\alpha). \quad (11)$$

Remak. The norm of κ over $\mathbb{Q}$

$$\mathbf{N}_{\kappa/\mathbb{Q}} : \kappa \longrightarrow \mathbb{Q} \quad (12)$$

is a multiplicative homomorphism of κ_* into $\mathbb{Q}_*$, namely

$$\mathbf{N}_{\kappa/\mathbb{Q}}(\alpha\beta) = \mathbf{N}_{\kappa/\mathbb{Q}}(\alpha)\mathbf{N}_{\kappa/\mathbb{Q}}(\beta) \in \kappa_*, \quad \alpha, \beta \in \kappa_*.$$

Remak. The trace of κ over $\mathbb{Q}$

$$\mathbf{Tr}_{\kappa/\mathbb{Q}} : \kappa \longrightarrow \mathbb{Q} \quad (13)$$

determines a $\mathbb{Q}$-linear mapping of κ to $\mathbb{Q}$, namely, for $\alpha, \beta \in \kappa$, $a \in \mathbb{Q}$,

$$\mathbf{Tr}_{\kappa/\mathbb{Q}}(\alpha + \beta) = \mathbf{Tr}_{\kappa/\mathbb{Q}}(\alpha) + \mathbf{Tr}_{\kappa/\mathbb{Q}}(\beta),$$

and

$$\mathbf{Tr}_{\kappa/\mathbb{Q}}(a\alpha) = a\mathbf{Tr}_{\kappa/\mathbb{Q}}(\alpha).$$

Let $\mathcal{O}_\kappa$ be the integral closure of $\mathbb{Z}$ in κ, that is,

$$\mathcal{O}_\kappa = \bar{\mathbb{Z}} \cap \kappa. \quad (14)$$

There are n elements w_1, ..., w_n in $\mathcal{O}_\kappa$ such that if the x_i run through all elements of $\mathbb{Z}$ in the expression

$$\beta = x_1w_1 + x_2w_2 + \cdots + x_nw_n,$$

we obtain each element in $\mathcal{O}_\kappa$ exactly once, that is, w_1, ..., w_n is a basis of $\mathcal{O}_\kappa$. Consequently

$$D_{\kappa/\mathbb{Q}} = D_{\kappa/\mathbb{Q}}(w_1, ..., w_n) = \det(\mathbf{Tr}_{\kappa/\mathbb{Q}}(w_iw_j)) \quad (15)$$

is independent of the choice of basis and is determined completely by the field itself. The nonzero rational integer $D_{\kappa/\mathbb{Q}}$ is called the *discriminant of field* κ, .

2 Minkowski's inequality: Geometry of numbers

2.1 Minkowski's bound

Let κ be a number field of degree n. For $\kappa = \mathbb{Q}$, $D_{\kappa/\mathbb{Q}} = 1$, and one of Minkowski's fundamental results was the proof that $|D_{\kappa/\mathbb{Q}}| > 1$ for $n > 1$. He later obtained a low bound for $|D_{\kappa/\mathbb{Q}}|$ that was exponential in n:

Theorem 2.1 (Minkowski's bound)**.** *The discriminant of an algebraic number field κ of degree n satisfies*

$$\sqrt{|D_{\kappa/\mathbb{Q}}|} \geq \frac{n^n}{n!}\left(\frac{\pi}{4}\right)^{\frac{n}{2}}.$$

The Minkowski's bound numerically was

$$|D_{\kappa/\mathbb{Q}}|^{1/n} > (7.389)^{r_1/n}(5.803)^{2r_2/n} \tag{16}$$

when n is large.

2.2 Rogers-Mulholland's improvement

Rogers [10] and Mulholland [8] used geometry of numbers methods to obtain an lower bounds of $|D_{\kappa/\mathbb{Q}}|$

$$|D_{\kappa/\mathbb{Q}}|^{1/2} \geq \max_{m\geq 2} \frac{(2n)^n}{n!(m-1)}\left(\frac{2}{\sqrt{\pi}}c_m\right)^{-2r_2} k_m^{-r_1}, \tag{17}$$

where $c_m = \min\{c'_m, 2\}$, $k_m = \min\{k'_m, 2\}$,

$$
\begin{aligned}
k'_m &= \frac{\pi}{2\sqrt{e}} \left(\frac{e^3 \pi^2 m (m-1)^2}{16} \right)^{\frac{1}{2m-2}}, \\
c'_m &= \frac{1}{m-1} \left(\frac{2m}{\sqrt{e}} \right)^{\frac{m}{m-1}}.
\end{aligned}
$$

For large n, these estimates imply that

$$
|D_{\kappa/\mathbb{Q}}|^{1/n} > (32.561\cdots)^{r_1/n} (15.775\cdots)^{2r_2/n} + o(1) \qquad (18)
$$

as $n \to \infty$.

3 Stark's analytic method

3.1 Analytic methods: Dedekind ζ-function

Let κ be a number field of degree n. The different cosets in $\mathcal{O}_\kappa$ determined by a nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_\kappa$ form the different residue classes mod($\mathfrak{a}$). The number of distinct residue classes mod($\mathfrak{a}$) is the index $[\mathcal{O}_\kappa : \mathfrak{a}]$ of $\mathfrak{a}$ in $\mathcal{O}_\kappa$. This index is finite. The number of residue classes is denoted by $\mathcal{N}(\mathfrak{a})$, called the (*absolute*) *norm* of $\mathfrak{a}$, or is also called the *counting norm* .

The Dedekind ζ-function of the number field κ is defined by the series

$$
\zeta_\kappa(s) = \sum_{\mathfrak{a}} \frac{1}{\mathcal{N}(\mathfrak{a})^s}, \qquad (19)
$$

where $\mathfrak{a}$ varies over the non-zero integral ideals of κ. The Dedekind function $\zeta_\kappa(s)$ admits a holomorphic continuation with the exclusion of a simple pole at $s = 1$, and satisfies the following Hadamard's factorization

$$\xi_\kappa(s) = e^{a-bs}\prod_{n=1}^{\infty}\left(1-\frac{s}{\rho_n}\right)e^{\frac{s}{\rho_n}} \tag{20}$$

for two constants $a, b(>0)$, where ρ_n are zeros of $\xi_\kappa(s)$ satisfying the conditions: $0 \leq \mathrm{Re}(\rho_n) \leq 1$, and

$$\xi_\kappa(s) = \frac{s}{2}(s-1)|D_{\kappa/\mathbb{Q}}|^{s/2}\Gamma_\mathbb{R}(s)^{r_1}\Gamma_\mathbb{C}(s)^{r_2}\zeta_\kappa(s) \tag{21}$$

in which

$$\Gamma_\mathbb{R}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \tag{22}$$

$$\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s). \tag{23}$$

The *Generalized Riemann Hypothesis* (GRH) for κ is the conjecture that all the zeros of the zeta function $\zeta_\kappa(s)$ that lie within the *critical strip* $0 < \mathrm{Re}(s) < 1$ actually lie on the *critical line* $\mathrm{Re}(s) = 1/2$.

3.2 Stark's result

H. Stark [11], [12] introduced an analytic method for proving lower bounds for discriminants by showing that for every complex s (other than $0, 1$, or a zero of $\zeta_\kappa(s)$),

$$\begin{aligned}\log|D_{\kappa/\mathbb{Q}}| &= r_1\left(\log\pi - \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right) + 2r_2\left(\log(2\pi) - \frac{\Gamma'(s)}{\Gamma(s)}\right) \\ &\quad -\frac{2}{s} - \frac{2}{s-1} + 2{\sum_\rho}' \frac{1}{s-\rho} - 2\frac{\zeta_\kappa'(s)}{\zeta_\kappa(s)},\end{aligned} \tag{24}$$

where ρ runs over the zeros of $\zeta_\kappa(s)$ in the critical strip, and $\sum_\rho'$ means that the ρ and $\bar{\rho}$ terms are to be taken together. This identity is a variant of the classical identity that comes from the Hadamard factorization of $\xi_\kappa(s)$, and Stark noticed that two of the constants that occur in that identity and which are hard to estimate can be eliminated. If s is real, $s > 1$, the identity (24) immediately yields

$$\log|D_{\kappa/\mathbb{Q}}| \;>\; r_1\left(\log\pi - \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right) + 2r_2\left(\log(2\pi) - \frac{\Gamma'(s)}{\Gamma(s)}\right) - \frac{2}{s} - \frac{2}{s-1}. \tag{25}$$

By taking $s = 1 + n^{-1/2}$, say, one obtains from (25) the estimate

$$|D_{\kappa/\mathbb{Q}}|^{1/n} \geq (4\pi e^{\gamma})^{r_1/n}\,(2\pi e^{\gamma})^{2r_2/n} - o(1) \tag{26}$$

as $n \to \infty$, where $\gamma = 0.5772156\cdots$ denotes Euler's constant, and

$$4\pi e^{\gamma} = 22.3816\cdots,\; 2\pi e^{\gamma} = 11.1908\cdots.$$

This is better than Minkowski's original result:

$$|D_{\kappa/\mathbb{Q}}|^{1/n} > (7.389)^{r_1/n}(5.803)^{2r_2/n}.$$

4 Odlyzko's method

4.1 Improvement of analytic method

A. M. Odlyzko [9] introduced a new analytic method of estimating discriminants based on the identity (24) of Stark. Consider a differentiable function $F : \mathbb{R} \longrightarrow \mathbb{R}$ with $F(-x) = F(x)$, $F(0) = 1$, and such that for some constants $c, \varepsilon > 0$,

$$|F(x)|,\; |F'(x)| \leq ce^{-(1/2+\varepsilon)|x|} \tag{27}$$

as $x \to \infty$. Define

$$\Phi(s) = \int_{-\infty}^{\infty} F(x) e^{(s-1/2)x} dx. \tag{28}$$

Then the explicit formula for the discriminant states that

$$\begin{aligned}
\log |D_{\kappa/\mathbb{Q}}| &= \frac{\pi r_1}{2} + \{\gamma + \log(8\pi)\} n - n \int_0^{\infty} \frac{1 - F(x)}{2 \sinh(x/2)} dx \\
&\quad - r_1 \int_0^{\infty} \frac{1 - F(x)}{2 \cosh(x/2)} dx - 4 \int_0^{\infty} F(x) \cosh \frac{x}{2} dx \\
&\quad + 2 \sum_{\mathfrak{P}} \sum_{m=1}^{\infty} \frac{\log \mathcal{N}(\mathfrak{P})}{\mathcal{N}(\mathfrak{P})^{m/2}} F(m \log \mathcal{N}(\mathfrak{P})) \\
&\quad + {\sum_{\rho}}' \Phi(\rho), \qquad (29)
\end{aligned}$$

where $\mathfrak{P}$ runs over prime ideals of κ.

In order to obtain a lower bound for $|D_{\kappa/\mathbb{Q}}|$ from (29), one selects $F(x) \geq 0$ for all x and $\mathrm{Re}(\Phi(s)) \geq 0$ for all s in the critical strip, so that the contributions of the prime ideals and zeros are nonnegative. The above nonnegativity conditions on $F(x)$ and $\Phi(s)$ are equivalent to the requirement that

$$F(x) = \frac{f(x)}{\cosh(x/2)}, \tag{30}$$

where $f(x) \geq 0$ and $f(x)$ has nonnegative Fourier transform. The best currently known unconditional bounds are obtained by selecting $f(x) = g(x/b)$ for some parameter b (depending on r_1 and r_2), where $g(x)$ is a certain function constructed by

L. Tartar. With this choice A. M. Odlyzko [9] found that

$$\begin{aligned} |D_{\kappa/\mathbb{Q}}|^{1/n} &\geq \left(4\pi e^{1+\gamma}\right)^{r_1/n}(4\pi e^{\gamma})^{2r_2/n} - O(n^{-2/3}) \qquad (31) \\ &= (60.8395\cdots)^{r_1/n}(22.3816\cdots)^{2r_2/n} - O(n^{-2/3}) \end{aligned}$$

which is better than Rogers-Mulholland's result:

$$|D_{\kappa/\mathbb{Q}}|^{1/n} > (32.561\cdots)^{r_1/n}(15.775\cdots)^{2r_2/n} + o(1).$$

Further, A. M. Odlyzko [9] remarks that no other choice of $f(x)$ can give a lower bound for $|D_{\kappa/\mathbb{Q}}|^{1/n}$ that has a larger main term than Eq. (31).

4.2 Further result under GRH

When one assumes the Generalized Riemann Hypothesis for $\zeta_\kappa(s)$, much better results are possible. In this case one only needs $F(x) \geq 0$ such that the Fourier transform of $F(x)$ is nonnegative. There are many choice of $G(x)$ such that $F(x) = G(x/b)$ for a proper choice of the scaling parameter b gives the bound

$$|D_{\kappa/\mathbb{Q}}|^{1/n} \geq (8\pi e^{\gamma+\pi/2})^{r_1/n}(8\pi e^{\gamma})^{2r_2/n} - O(\log^{-2} n), \qquad (32)$$

where

$$\begin{aligned} 8\pi e^{\gamma+\pi/2} &= 215.3325\cdots, \\ 8\pi e^{\gamma} &= 44.7632\cdots. \end{aligned}$$

Just as in the unconditional case, no choice of $F(x)$ can give a better main term (see [9]). In Open Problem 6.3, A. M. Odlyzko [9] suggested the following question:

Question 4.1. *Are the GRH bounds for discriminants valid even without the assumption of the GRH?*

5 Our work

5.1 Main result

The Dedekind function $\zeta_\kappa(s)$ satisfies the following functional equation

$$\zeta_\kappa(1-s) = A(s)\zeta_\kappa(s), \tag{33}$$

where

$$A(s) = |D_{\kappa/\mathbb{Q}}|^{s-\frac{1}{2}} \left(\cos\frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin\frac{\pi s}{2}\right)^{r_2} 2^{(1-s)n}\pi^{-sn}\Gamma^n(s).$$

A straightforward computation gives $A(1/2) = 1$.

Let α_κ $(\neq 0)$, β_κ be defined by the Taylor expansion of $\zeta_\kappa(s)$ at $s = 1/2$, i.e.,

$$\zeta_\kappa(s) = \alpha_\kappa \left(s - \frac{1}{2}\right)^\mu + \beta_\kappa \left(s - \frac{1}{2}\right)^{\mu+1} + \cdots . \tag{34}$$

It turns out from (33) that μ is a non-negative even integer. In this short paper, we show an identity which expresses the discriminant in terms of the ratio of the first two coefficients of the Taylor series of ζ_κ at $1/2$. It follows that the main term in inequality (32) would be achieved without assuming the Generalized Riemann Hypothesis.

Theorem 5.1 ([6]). *Let κ be a number field of degree n and let $D_{\kappa/\mathbb{Q}}$, r_1, r_2, α_κ and β_κ be defined as above. Then, we have*

$$\begin{aligned} |D_{\kappa/\mathbb{Q}}|^{1/n} &= (8\pi e^{\gamma+\pi/2})^{r_1/n} (8\pi e^{\gamma})^{2r_2/n} e^{-2\beta_\kappa/(n\alpha_\kappa)} \qquad (35) \\ &= (215.3325\cdots)^{r_1/n} (44.7632\cdots)^{2r_2/n} e^{-2\beta_\kappa/(n\alpha_\kappa)}. \end{aligned}$$

Proof. It is clear from (34) that

$$\frac{\zeta_\kappa'(s)}{\zeta_\kappa(s)} = \frac{\mu}{s-\frac{1}{2}} + \frac{\beta_\kappa}{\alpha_\kappa} + O\left(\left(s-\frac{1}{2}\right)\right)$$

and

$$-\frac{\zeta_\kappa'(1-s)}{\zeta_\kappa(1-s)} = \frac{\mu}{s-\frac{1}{2}} - \frac{\beta_\kappa}{\alpha_\kappa} + O\left(\left(s-\frac{1}{2}\right)\right).$$

Logarithmically differentiating (33) gives

$$-\frac{\beta_\kappa}{\alpha_\kappa} = \log|D_{\kappa/\mathbb{Q}}| - \frac{r_1\pi}{2} - n\log(2\pi) + n\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + \frac{\beta_\kappa}{\alpha_\kappa}.$$

By using the fact (e.g. [5], page 482) that

$$\Gamma'\left(\frac{1}{2}\right)/\Gamma\left(\frac{1}{2}\right) = -\gamma - \log 4,$$

we obtain

$$\frac{\beta_\kappa}{\alpha_\kappa} = \frac{n}{2}\{\gamma + \log(8\pi)\} + \frac{r_1\pi}{4} - \frac{1}{2}\log|D_{\kappa/\mathbb{Q}}|,$$

which completes the proof. □

Clearly, the theorem provides us with a formula to compute the discriminant of individual number fields. Further, it's worth to point out that the main term in (32) appears exactly in the theorem. Thus, the inequality (32) can be proved without assuming the Generalized Riemann Hypothesis if one can give an absolute upper bound for $\beta_\kappa/\alpha_\kappa$.

5.2 $\kappa = \mathbb{Q}$

When $\kappa = \mathbb{Q}$, we know (see [3], [4]):

$$\frac{\beta_\kappa}{\alpha_\kappa} = \frac{\zeta'\left(\frac{1}{2}\right)}{\zeta\left(\frac{1}{2}\right)} = \frac{\pi}{4} + \frac{\gamma}{2} + \frac{1}{2}\log 8\pi. \tag{36}$$

5.3 $\kappa = \mathbb{Q}(\sqrt{d})$

We have

$$\zeta_\kappa(s) = \zeta(s)L(s),$$

where

$$L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

in which

$$\chi(n) = \left(\frac{d}{n}\right).$$

We have

$$L(1-s) = \frac{2}{i^\delta \tau(\chi)} \left(\frac{2\pi}{r}\right)^{-s} \cos\left(\frac{\pi(s+\delta)}{2}\right) \Gamma(s)L(s), \tag{37}$$

where $r = |d|$, $\tau(\chi)$ is the *Gauss sum*

$$\tau(\chi) = \sum_{n=0}^{r-1} \chi(n) e^{\frac{2\pi i n}{r}}$$

with $|\tau(\chi)| = \sqrt{r}$, and

$$\delta = \begin{cases} 0, & \text{if } d > 0; \\ 1, & \text{if } d < 0. \end{cases}$$

Therefore

$$-\frac{L'(1-s)}{L(1-s)} = -\log\frac{2\pi}{r} - \frac{\pi}{2}\tan\left(\frac{\pi(s+\delta)}{2}\right) + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{L'(s)}{L(s)}.$$

Note that

$$\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = -\gamma - \log 4.$$

We obtain

$$\begin{aligned}\frac{L'\left(\frac{1}{2}\right)}{L\left(\frac{1}{2}\right)} &= \frac{1}{2}\log\frac{2\pi}{r} + \left(\frac{1}{2}-\delta\right)\frac{\pi}{2} - \frac{1}{2}\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{1}{2}\log\frac{2\pi}{r} + \left(\frac{1}{2}-\delta\right)\frac{\pi}{2} + \frac{1}{2}(\gamma + \log 4).\end{aligned}$$

Thus

$$\frac{\beta_\kappa}{\alpha_\kappa} = \frac{\zeta'_\kappa\left(\frac{1}{2}\right)}{\zeta_\kappa\left(\frac{1}{2}\right)} = (1-\delta)\frac{\pi}{2} + \gamma + \log 8\pi - \frac{1}{2}\log|d|.$$

5.4 A bound of $\frac{\beta_\kappa}{\alpha_\kappa}$

By the main theorem, we obtain a bound

$$\frac{\beta_\kappa}{\alpha_\kappa} \leq \left(\frac{\pi}{2} + \gamma + \log 8\pi\right)\frac{n}{2}$$

since $|D_{\kappa/\mathbb{Q}}| \geq 1$.

If the Generalized Riemann Hypothesis is assumed, the inequality (32) yields

$$
\begin{aligned}
\frac{1}{n}\log|D_{\kappa/\mathbb{Q}}| &\geq \left(\frac{\pi}{2}+\gamma+\log 8\pi\right)\frac{r_1}{n}+(\gamma+\log 8\pi)\frac{2r_2}{n} \\
&\quad +\log\left(1-O(\log^{-2} n)\right) \\
&\geq \left(\frac{\pi}{2}+\gamma+\log 8\pi\right)\frac{r_1}{n}+(\gamma+\log 8\pi)\frac{2r_2}{n} \\
&\quad -O(\log^{-2} n)
\end{aligned}
$$

as $n \to \infty$, and hence

$$\frac{\beta_\kappa}{\alpha_\kappa} \leq O\left(\frac{n}{\log^2 n}\right).$$

6 Another relation

6.1 A result on GRH

Take R with $R > 0$ and set

$$\mathbb{C}(0;R) = \{z \in \mathbb{C} \mid |z| < R\},\ \mathbb{C}[0;R] = \{z \in \mathbb{C} \mid |z| \leq R\}.$$

Now we use the *Carleman's formula* (cf. [5]):

Lemma 6.1. *Let $f(z)$ be meromorphic in $\mathbb{C}[0;R] \cap \{\mathrm{Re}(z) \geq 0\}$ with $f(0) = 1$, and suppose that it has the zeros $r_1e^{i\theta_1}$, $r_2e^{i\theta_2}$, ..., $r_me^{i\theta_m}$ and the poles $s_1e^{i\varphi_1}$, $s_2e^{i\varphi_2}$, ..., $s_ne^{i\varphi_n}$ inside $\mathbb{C}(0;R) \cap \{\mathrm{Re}(z) > 0\}$. Then*

$$
\begin{aligned}
&\sum_{\mu=1}^{m}\left(\frac{1}{r_\mu}-\frac{r_\mu}{R^2}\right)\cos\theta_\mu-\sum_{\nu=1}^{n}\left(\frac{1}{s_\nu}-\frac{s_\nu}{R^2}\right)\cos\varphi_\nu \\
=\ &\mathcal{C}_f(R)-\frac{1}{2}\mathrm{Re}(f'(0)),
\end{aligned}
$$

where

$$\mathcal{C}_f(R) = \frac{1}{\pi R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|f\left(Re^{i\theta}\right)\right|\cos\theta d\theta$$
$$+\frac{1}{2\pi}\int_0^R \left(\frac{1}{y^2}-\frac{1}{R^2}\right)\log|f(iy)f(-iy)|dy. \qquad (38)$$

Theorem 6.2. *Generalized Riemann Hypothesis is true if and only if*

$$\lim_{R\to\infty}\mathcal{C}_f(R) = \frac{\beta_\kappa}{2\alpha_\kappa} - 2,$$

where γ *is Euler's constant, and*

$$f(s) = \frac{\zeta_\kappa\left(s+\frac{1}{2}\right)}{\alpha_\kappa s^\mu}. \qquad (39)$$

Proof. Note that

$$z_\mu - \frac{1}{2} = r_\mu e^{i\theta_\mu}\ \left(r_\mu > 0,\ 0 < \theta_\mu < \frac{\pi}{2}\right),\ \overline{z}_\mu - \frac{1}{2}$$

would be the zeros of f in the half-plane $\mathrm{Re}(s) > 0$, and $s = \frac{1}{2}$ is the unique pole of f in $\mathrm{Re}(s) > 0$. Hence Lemma 6.1 implies

$$2\sum_{r_\mu<R}\left(\frac{1}{r_\mu}-\frac{r_\mu}{R^2}\right)\cos\theta_\mu - \left(2-\frac{1}{2R^2}\right) = \mathcal{C}_f(R) - \frac{1}{2}\mathrm{Re}(f'(0)).$$

Since f is of order 1, then the convergence exponent of zeros for f is at most 1. Hence the series

$$\sum_\mu \frac{1}{r_\mu^{1+\varepsilon}}$$

is convergent for any $\varepsilon > 0$, and so

$$0 \le \sum_\mu \frac{\cos\theta_\mu}{r_\mu} = \sum_\mu \frac{r_\mu\cos\theta_\mu}{r_\mu^2} \le \frac{1}{2}\sum_\mu\frac{1}{r_\mu^2} < \infty.$$

Let $N(R)$ count the number of zeros ρ of $\zeta_\kappa(s)$ satisfying $\mathrm{Re}(\rho) > 0$, $|\mathrm{Im}(\rho)| \le R$. By Theorem 7.7 of [13], we find

$$N(R) = \frac{n}{\pi} R \log \frac{R}{e} + \frac{R}{\pi} \log \frac{|D_{\kappa/\mathbb{Q}}|}{(2\pi)^n} + O(\log R),$$

and hence

$$0 \le \sum_{r_\mu < R} \frac{r_\mu \cos\theta_\mu}{R^2} \le \frac{N(R)}{2R^2} \to 0 \ (R \to \infty).$$

Thus we obtain

$$\lim_{R\to\infty} \mathcal{C}_f(R) = 2 \sum_\mu \frac{\cos\theta_\mu}{r_\mu} + \frac{\beta_\kappa}{2\alpha_\kappa} - 2.$$

Generalized Riemann Hypothesis is true if and only if the zeros z_μ do not exist, that is,

$$\sum_\mu \frac{\cos\theta_\mu}{r_\mu} = 0,$$

equivalently,

$$\lim_{R\to\infty} \mathcal{C}_f(R) = \frac{\beta_\kappa}{2\alpha_\kappa} - 2.$$

$\square$

6.2 Estimate on the integral $\mathcal{C}_f(R)$

Next we make a remark on the integral $\mathcal{C}_f(R)$ in Theorem 6.2. We know (cf. [2])

$$\int_0^R |\zeta_\kappa(\sigma + it)|^2 \, dt \le O\left(R^{n(1-\sigma)} (\log R)^n\right)$$

for $\frac{1}{2} \leq \sigma \leq 1-\frac{1}{n}$. By the concavity of the logarithmic function, we obtain

$$\begin{aligned} \frac{1}{R^2}\int_0^R \log|f(it)|^2 dt &\leq \frac{1}{R}\log\left\{\frac{1}{R}\int_0^R |f(it)|^2 dt\right\} \\ &= O\left(\frac{\log R}{R}\right). \end{aligned} \tag{40}$$

We also know (see [13], Theorem 6.8)

$$\limsup_{t\to\pm\infty}\frac{\log|\zeta_\kappa(\sigma+\mathrm{i}t)|}{\log|t|} \leq \begin{cases} 0, & \text{if } \sigma > 1; \\ \frac{n}{2}(1-\sigma), & \text{if } 0 \leq \sigma \leq 1; \\ \left(\frac{1}{2}-\sigma\right)n, & \text{if } \sigma < 0, \end{cases}$$

which yield immediately

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\zeta_\kappa\left(\frac{1}{2}+Re^{i\theta}\right)\right|\cos\theta d\theta \leq O(\log R). \tag{41}$$

By careful computations, we can obtain converse inequalities of (40) and (41). Therefore from Theorem 6.2, (40) and (41), we can obtain:

Theorem 6.3 ([7])**.**

$$\lim_{R\to\infty}\mathcal{C}_f(R) = \frac{1}{2\pi}\int_0^\infty \log|f(it)|^2\frac{dt}{t^2}.$$

Since $f(s)$ is defined by (39), we immediately get

Corollary 6.4 ([7])**.** *Generalized Riemann Hypothesis is true if and only if*

$$\frac{1}{\pi}\int_0^\infty t^{-2}\log\frac{\left|\zeta_\kappa\left(\frac{1}{2}+it\right)\right|^2}{|\alpha_\kappa|^2 t^{2\mu}}dt = \frac{\beta_\kappa}{\alpha_\kappa}-4.$$

The formula (36) yields immediately the following fact:

Corollary 6.5 ([3]). *Riemann Hypothesis is true if and only if*

$$\frac{1}{\pi}\int_0^\infty t^{-2}\log\left|\frac{\zeta\left(\frac{1}{2}+it\right)}{\zeta\left(\frac{1}{2}\right)}\right|^2 dt = \frac{\pi}{4}+\frac{\gamma}{2}+\frac{1}{2}\log 8\pi - 4.$$

Note that the identity (35) is equivalent to the identity

$$\log|D_{\kappa/\mathbb{Q}}| = \{\gamma+\log(8\pi)\}n + \frac{\pi}{2}r_1 - 2\frac{\beta_\kappa}{\alpha_\kappa}. \tag{42}$$

Therefore, by Theorem 6.2, Theorem 6.3 and (42), we have

Theorem 6.6 ([7]). *Generalized Riemann Hypothesis for $\zeta_\kappa(s)$ is true if and only if*

$$\begin{aligned}\log|D_{\kappa/\mathbb{Q}}| &= \{\gamma+\log(8\pi)\}n + \frac{\pi}{2}r_1 \\ &\quad -8 - \frac{2}{\pi}\int_0^\infty t^{-2}\log\frac{\left|\zeta_\kappa\left(\frac{1}{2}+it\right)\right|^2}{|\alpha_\kappa|^2 t^{2\mu}}dt. \end{aligned} \tag{43}$$

6.3 A note on zeros of $\zeta_\kappa(s)$

Armitage [1] constructed a field κ for which $\zeta_\kappa\left(\frac{1}{2}\right) = 0$. In Open Problem 6.2, A. M. Odlyzko [9] suggested the following question:

Question 6.7. *Do the zeros of $\zeta_\kappa(s)$ in the critical strip approach the real axis as $n \to \infty$, and if they do, how fast do they do so, and how many of them are there?*

The best results appear to be due to Odlyzko, and show that on the GRH, $\zeta_\kappa(s)$ has a zero on the critical line at height $O((\log n)^{-1})$ as $n \to \infty$. Unconditionally, it has only been shown that there is a zero at height $\leq 0.54 + o(1)$ as $n \to \infty$, and that for every κ with $n \geq 2$, there is a zero at height < 14. The first zero of the Riemann zeta function is at height $14.1347\cdots$, so this result shows that the zeta function is extremal in terms of having its lowest zero as high as possible.

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Department of Mathematics, Shandong University, Jinan 250100, Shandong, P. R. China

e-mail: pchu@sdu.edu.cn